Instability in Spatial Evolutionary Games

Carlos Grilo[1,2] and Luís Correia[2]

[1] Dep. Eng. Informática, Escola Superior de Tecnologia e Gestão, Instituto Politécnico de Leiria Portugal

[2] LabMag, Dep. Informática, Faculdade Ciências da Universidade de Lisboa, Portugal

grilo@estg.ipleiria.pt, luis.correia@di.fc.ul.pt

Abstract. We investigate the aspects that influence the instability of spatial evolutionary games, namely the Prisoner's Dilemma and the Snowdrift games. In this paper instability is defined as the proportion of strategy changes in the asymptotic period of the evolutionary process. The results show that with the Prisoner's Dilemma, when the level of noise present in the decision process is very low, the instability decreases as the synchrony rate decreases. With the Snowdrift this pattern of behavior depends strongly on the interaction topology and arises only for random and scale-free networks. However, for large noise values, the instability in both games depends only on the proportion of cooperators present in the population: it increases as the proportion of cooperators approaches 0.5. We advance an explanation for this behavior.

1 Introduction

Spatial evolutionary games are used as models to study, for example, how cooperation could ever emerge in nature and human societies [14]. They are also used as models to study how cooperation can be promoted and sustained in artificial societies [12]. In these models, a structured population of agents interacts during several time steps through a given *game* which is used as a metaphor for the type of interaction that is being studied. The population is structured in the sense that each agent can only interact with its neighbors. The underlying structure that defines who interacts with whom is called the *interaction topology*. After each interaction session, some or all the agents, depending on the *update dynamics* used, have the possibility of changing their strategies. This is done using a so called *transition rule* that models the fact that agents tend to adapt their behavior to the context in which they live by imitating the most successful agents they know. It can also be interpreted as the selection step of an evolutionary process in which the least successful strategies tend to be replaced by the most successful ones.

Spatial evolutionary games were created by Martin Nowak and Robert May in a seminal work in 1992 [11]. They showed that cooperation can be maintained when the Prisoner's Dilemma game is played on a regular 2-dimensional grid. This is not possible in well-mixed populations, in which each agent can play the game with any other player in the population. Since then, a large number of works have been published about how different input conditions influence both the level of cooperation and the evolutionary dynamics. For example, in [13] and [7] the influence of the interaction topology is examined. Also, in [16] the influence of the interaction topology, the transition rule and the update dynamics in the Hawk-Dove or Snowdrift game are studied. The influence of the update dynamics is addressed, for example, in [8] and [9]. We also studied the influence of the update dynamics and found that, in general an asynchronous dynamics supports more cooperators than a synchronous one [3, 4, 5, 6].

In this paper we address the problem of how different conditions influence the instability of the system. That is, once the system enters an asymptotic phase, where the proportion of cooperators in the population stabilizes, we are interested in knowing how much agents change their strategies, from cooperation to defection and vice-versa and what are the factors that influence this phenomenon. The proportion of surviving cooperators is undoubtedly a relevant measure of interest and this is why it has captured almost all the attention in previous works. However, the proportion of cooperators in an asymptotic phase rarely is a constant value: it varies within an interval. Even if nearly constant, it may be the result of agents changing strategies from cooperator to defector an the other way around. Instability is an important measure because a volatile society, where agents often change their strategies, turns it difficult to identify defecting agents and, therefore, to undertake measures for the promotion of cooperation. It can also be interpreted as a measure of the level of dissatisfaction in the population since unsatisfied agents tend to change their way of acting. As far as we know, until now instability was only studied by Abramson and Kuperman [1]. They studied the Prisoner's Dilemma game played on small-world networks under the best-neighbor transition rule (see Section 2). They found that instability grows as the interaction topology approaches a random network.

We apply the study of instability to the Prisoner's Dilemma and the Snowdrift games, which are the two most used games in this area. There are at least two good reasons to use more than one game in this type of investigation. The first one is that both the similarities and the differences in the results achieved with different games can lead us to a better understanding of the problem. The second reason follows from the difficulty that field researchers frequently experience in the evaluation of the relative value of the payoffs involved in concrete real situations [7]. Different payoff relations may define different games and that reinforces the need to experiment with several games. Frequently in this area, conclusions are driven based only on a limited number of tested conditions. Here, we use a broad number of conditions so that a better identification of how each aspect influences the behavior of the model is achieved. Namely, different interaction topologies, noise levels and synchrony rates are used.

The paper is structured as follows: in Section 2 we describe the model used in our experiments and the experimental setup. In Section 3 we present and discuss the results and, finally, in Section 4 some conclusions are drawn and future work is advanced.

2 The Model

2.1 The Games

The Prisoner's Dilemma (PD) and the Snowdrift (SD) are two-player games in which players can only cooperate (C) or defect (D). The payoffs are the following: R to each player if they both play C; P to each if they both play D; T and S if one plays D and the other C, respectively. These games differ in the relations existing between the payoff values: while in the PD game these values must obey $T > R > P > S$, in the SD game they must obey $T > R > S > P$. Given these conditions, it follows that, in the PD game, D is the best action to take regardless of the opponent's decision. In the SD the best action depends on the opponent's decision: the best thing to do is to take the opposite action the opponent takes. For practical reasons, it is common to rescale the payoffs such that the games can be described by one parameter only [13]. The PD's payoffs are defined as $R = 1$, $T = b > 1$ and $S = P = 0$, where b represents the advantage of D players over C ones when they play the game with each other. It was shown that with this payoffs the game keeps its essential properties [11]. In the SD game, payoffs are defined as $T = \beta \geq 1$, $R = \beta - 1/2$, $S = \beta - 1$ and $P = 0$ which leads to a cost-to-benefit ratio of mutual cooperation $r = 1/(2\beta - 1), 0 \leq r \leq 1$.

2.2 Interaction Topology

We used two types of interaction topologies: *small-world networks* (SWNs) [17] and *scale-free networks* (SFNs) [2]. In order to build SWNs, first a toroidal regular 2D grid is built so that each node is linked to its 8 surrounding neighbors by undirected links; then, with probability ϕ, each link is replaced by another one linking two randomly selected nodes. We do not allow self or repeated links nor disconnected graphs. Networks built this way have the property that, even for very small ϕ values, the average path length is much smaller than in a regular network, maintaining a high clustering coefficient. Both these properties are very commonly observed in real social systems. As $\phi \to 1$, we get random networks with both small average path lengths and clustering coefficients.

SFNs have a power law degree distribution $P(k) \sim k^{-\gamma}$ that is also very common in real social networks[3]. SFNs are built in the following way: the network is initialized with m fully connected nodes. Then, nodes are added, one at a time, until the network has the desired size. Each added node is linked to m already existing nodes so that the probability of creating a link with some existing node i is equal to $\frac{k_i}{\sum_j k_j}$, where k_i is the degree of i, that is, the number of nodes to which it is connected. After t time steps, the network has $m+t$ nodes and mt links (plus the links connecting the initial set of nodes). The resulting γ exponent is approximately 2.9 and is independent of m in large networks. This method of link creation is called *preferential attachment*, since the more links a node has, the greater is the probability of creating links to it. This has the effect

[3] $P(k)$ is the probability of a node with k neighbors.

that a small proportion of nodes has a big connectivity while a large majority has a very low connectivity.

2.3 Interaction and Strategy Update Dynamics

On each time step, agents first play a one round PD or SD game with all their neighbors. Agents are pure strategists which can only play C or D. After this interaction stage, each agent updates its strategy with probability α using a *transition rule* (see next section) that takes into account the payoff of the agent's neighbors. The update is done synchronously by all the agents selected to engage in this revision process. The α parameter is called the *synchrony rate* and is the same for all agents. When $\alpha = 1$ we have a synchronous model, where all the agents update at the same time. As $\alpha \to \frac{1}{n}$, where n is the population size, the model approaches sequential dynamics, where exactly one agent updates its strategy at each time step.

2.4 The Strategy Update Process

The strategy update process is done using a transition rule that models the fact that agents tend to imitate the most successful agents they know. In this work, we used the generalized *proportional* transition rule (GP) proposed in [10]. Let G_x be the average payoff earned by agent x, N_x be the set of neighbors of x and c_x be equal to 1 if x's strategy is C and 0 otherwise. According to this rule, the probability that an agent x adopts C as its next strategy is

$$p_C(x, K) = \frac{\sum_{i \in N_x \cup x} c_i (G_i)^{1/K}}{\sum_{i \in N_x \cup x} (G_i)^{1/K}}, \tag{1}$$

where $K \in]0, +\infty[$ can be viewed as the noise present in the strategy update process. Noise is present in this process if there is some possibility that an agent imitates strategies other than the one used by its most successful neighbor. Small noise values favor the choice of the most successful neighbors' strategies. Also, as noise diminishes, the probability of imitating an agent with a lower payoff becomes smaller. When $K \to 0$ we have a deterministic best-neighbor rule such that i always adopts the best neighbor's strategy. When $K = 1$ we have a simple proportional update rule. Finally, for $K \to +\infty$ we have random drift where payoffs play no role in the decision process. Our analysis considers the interval $K \in]0, 1]$.

2.5 Experiments Setup and Instability Measure

All the experiments were performed with populations of $50 \times 50 = 2500$ agents, randomly initialized with 50% of cooperators (Cs) and 50% of defectors (Ds). When the system is running synchronously, i.e., when $\alpha = 1$, we let it first run during a transient period of 900 iterations. After this, we let the system run for 100 more iterations during which the proportion of cooperators varies around a value that depends on the input conditions. During this asymptotic period, we measure the instability of the system as described in the next paragraph. When $\alpha \neq 1$ the number of selected agents at each time step may not be equal to the size of the population and it may vary between two consecutive time steps. In order to guarantee that these runs are equivalent to the synchronous ones in what concerns the total number of individual updates, we let the system first run until 900×2500 individual updates have been done. After this, we let the system run during more 100×2500 individual updates. Each experiment is a combination of b or r (for the PD and the SD, respectively), ϕ or m (for SWNs and SFNs, respectively), α, and K parameters. All the possible combinations of the values shown in Table 1 were tested. For each combination, 30 runs were made and the average instability values of these runs is taken as the output, as well as the standard-deviation. For each run, a different random number seed is used, including the generation of the interaction topology.

Parameter	**Values**
b (PD)	1, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 2
r (SD)	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1
ϕ (SWNs)	0 (regular), 1 (random), SW: 0.01, 0.05, 0.1
m (SFNs)	2, 4, 8
α	0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1
K	0, 1/100, 1/10, 1/8, 1/6, 1/4, 1/2, 1

Table 1. Parameter values used in the experiments.

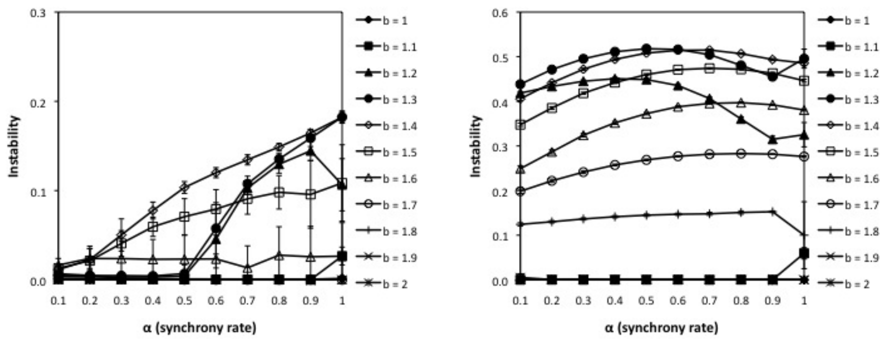
In this paper we consider the instability of the system as a measure of how much agents change their strategies in the asymptotic period. The instability of the system is, therefore, measured as follows:

$$\text{Instability} = \frac{\text{Number of strategy changes in the asymptotic period}}{\text{Number of individual updates in the asymptotic period}}, \quad (2)$$

where a strategy change means that an agent changed its strategy from C to D or vice-versa. Instability could be interpreted, instead, as the level of variation of the proportion of cooperators present in the population. We have also considered this in the experimental work by measuring the average standard-deviation of the proportion of cooperators. All the values are, however, very small, never exceeding 0.05 and rarely exceeding 0.01, which means that the level of cooperation is very stable in the asymptotic period.

3 Results

The first observation that can be made from the results is that the noise level, K, has not a direct consistent influence on the instability of the model. By



(a) (b)

Fig. 1. Instability as a function of α when the PD (a) and the SD (b) are played without noise ($K = 0$) on SWNs ($\phi = 0.1$).

this we mean that, other parameters being constant, we do not observe a direct dependency of instability on the noise level. However, there is a kind of second order type of influence. We observe that in the PD game, when the noise level is very small ($K = 0$ and $K = \frac{1}{100}$), the instability decreases as the synchrony rate (α) decreases. Saying it another way, the model becomes more stable, or less dynamic, as α decreases. Fig. 1(a) shows an example of this for the PD game. This pattern is common to all the interaction topologies we experimented with.

In the SD game, this pattern is, however, strongly dependent on the interaction topology: on SFNs, instability slightly increases as α is changed from 1.0 to 0.8. Then, from $\alpha = 0.8$ to $\alpha = 0.1$ it decreases, as in the PD game. On networks generated with the SWNs model, instability decreases with α only when $\phi = 1$ (random networks). As ϕ is decreased it becomes more difficult to identify a pattern of influence of the synchrony rate. Fig. 1b) shows an example for $\phi = 0.1$, where one can see that for some r values the instability decreases with α and that it increases for others.

The results described above apply only to small noise values. For larger noise values the synchrony rate plays no role on the instability of the model. Instead, as the noise level is increased, it turns out that instability becomes progressively dependent on the proportion of cooperators. More specifically, with large noise values, instability grows as the proportion of Cs approaches 0.5. This pattern is common to both games and is not dependent on the interaction topology that is used. Figs. 2 and 3 shows examples of this behavior for the PD and SD games, respectively.

This result means that for noise values close to 1, the input parameters seem to play no direct influence on the instability of the model. We can talk only about an indirect influence in the sense that both b/r, ϕ/m, K and α influence the level of cooperation, which in turn determines how unstable are these games

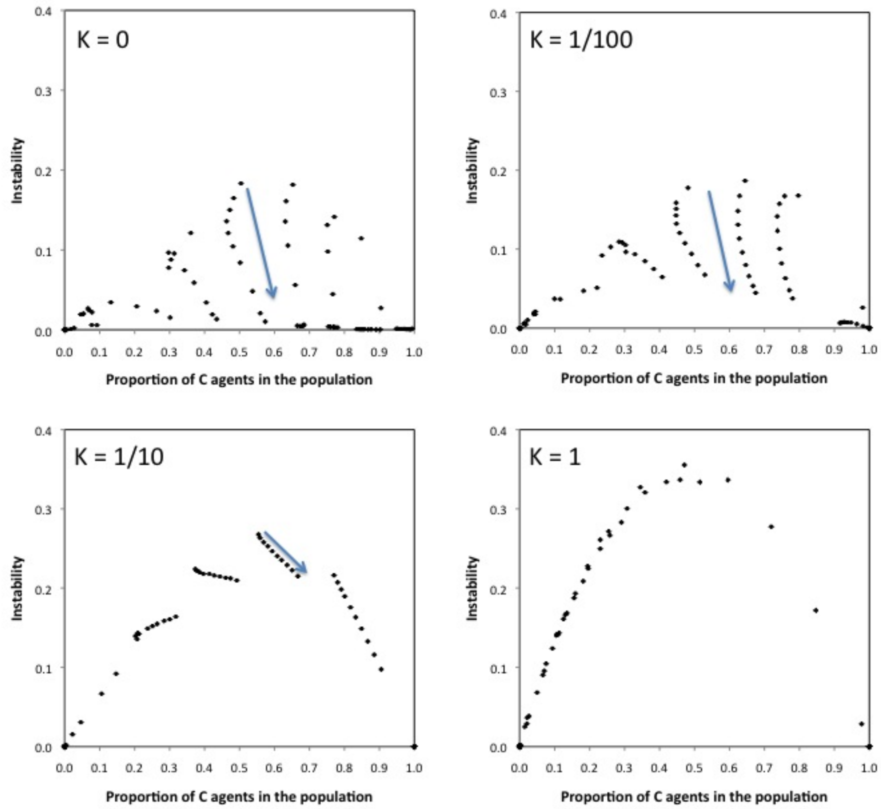


Fig. 2. Instability as a function of the proportion of cooperators in the population for different noise (K) values in the PD game played on SWNs ($\phi = 0.1$). In the charts corresponding to $K = 0, \frac{1}{100}, \frac{1}{10}$ and 1, one may observe that some points organize into line-like shapes. Each "line" corresponds to the instability values obtained for a given b value. The arrows point the direction of decreasing α. The chart for $K = 0$, corresponds to the situation of Fig. 1(a).

in the asymptotic period. It remains, however, to explain why, for K values near 1, the instability grows as the proportion of Cs approaches 0.5 .

As a starting point, we may look for situations in which the asymptotic proportion of Cs is close to 0. In these situations, there is a large majority of D-D links, and a small number of C-C and C-D links. Given that a C agent can only change its strategy to D if it has at least one D neighbor and vice-versa, this explains why the average number of strategy changes is small in these situations. The same reasoning can be done for situations in which the proportion of C agents is very close to 1. As the proportion of Cs approaches 0.5, it is expectable that the number of C-D links per agent gets larger on average.

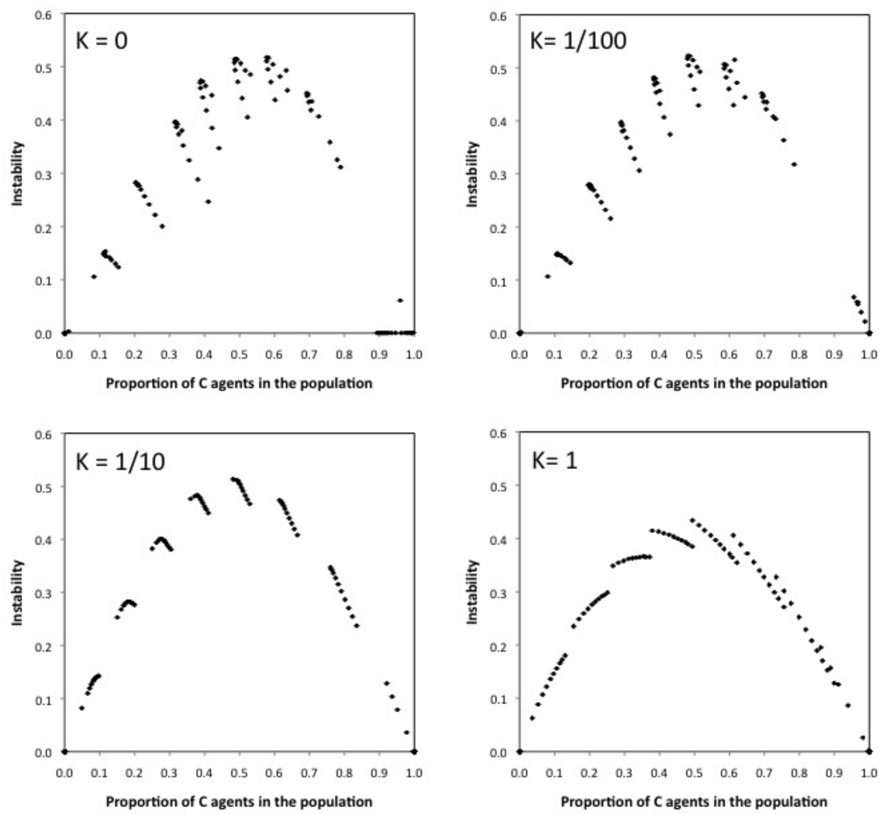


Fig. 3. Instability as a function of the proportion of cooperators in the population for different noise (K) values in the SD game played on SWNs ($\phi = 0.1$). The chart for $K = 0$, corresponds to the situation of Fig. 1(b).

In complete networks, where each agent is connected to all the other agents of the population, it is straightforward to compute the proportion of C-D links in the population. It is equal to

$$p_{CD}(n, k) = \frac{k(n-k)}{\sum_{i=1}^{n-1} i}, \tag{3}$$

where n is the size of the population and k is the number of C agents. In structured networks the value of $p_{CD}(n, k)$ depends on the shape of the network and on how the agents are distributed over the nodes. We numerically computed the average value of $p_{CD}(n, k)$ for the networks used in this work as described next. First, an interaction topology was generated. SWNs and SFNs were used, respectively with the ϕ and m values described in Table 1. Then, for each interaction topology all the possible configurations of the population were generated

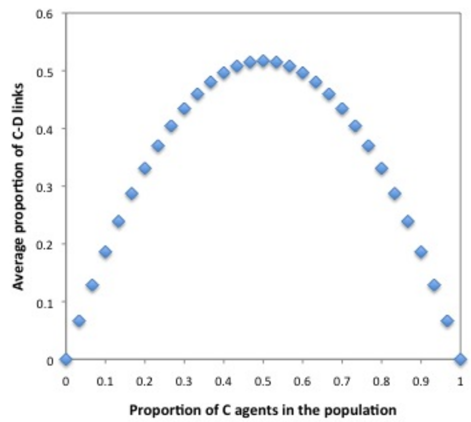
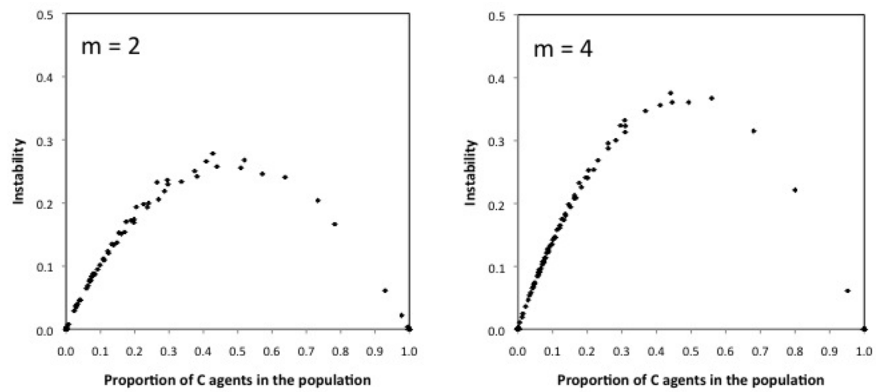


Fig. 4. Average proportion of C-D links per agent as a function of the proportion of Cs in the population over all possible configurations of 36 agents. The shape of the curve is exactly the same for complete networks.

(populations of 36 agents were used). For each configuration we measured the proportion of C-D links between agents and their neighbors. That is, for each possible configuration, we measured the proportion of links per agent that may lead to strategy changes. Notice that these were static populations. In this setting, we were only interested in knowing how the proportion of C agents in the population influences the potential for strategy changes (proportion of C-D links). The resulting chart in the Fig. 4 shows the average proportion of C-D links as a function of the proportion of Cs in the population. It shows that the average proportion of C-D links grows as the proportion of C agents approaches 0.5, exactly as for complete networks. The shape of the curve does not depend on the type of interaction topology. This curve strongly resembles the instability curves of both games when $K = 1$ (Figs. 2 and 3), specially for the PD game. This is an additional evidence that, in this regime, the instability depends mainly on the proportion of C agents in the population.

In order to understand why this pattern is absent for very small noise values and why it arises as the noise level gets larger, we must examine the GP transition rule and how K influences agents' decisions. As we have seen in Section 2.5, when $K \approx 0$, an agent always imitates the strategy of its wealthiest neighbor. In this regime there is not a direct correlation between the imitated strategy and the most common strategy in the neighborhood of an agent. For example, a C agent that has 7 C neighbors and 1 D neighbor will imitate the D one if this one has the highest payoff. However, as K approaches 1, the difference between the payoffs (powered to $\frac{1}{K}$) becomes smaller. This means that, as the noise level grows, the relative proportion of Cs and Ds in the neighborhood becomes an important factor in the decision process. For example, if there are more C agents in the



(a) (b)

Fig. 5. Instability as a function of the proportion of cooperators in a population PD players. Conditions: a) SFNs ($m = 2$), $K = 1$; b) SFNs ($m = 4$), $K = 1$.

neighborhood than D agents, the sum of their payoffs may exceed the sum of the D payoffs even if the C agents have smaller payoffs individually. This does not guarantees that the strategy with the larger payoff sum will be imitated since the stochastic nature of the rule allows less successful strategies to be selected when $K \neq 0$. However, as K approaches 1, the decision process will tend to reflect more and more the composition of the neighborhood. The effect it impinges on the instability is the following: when there is a clear difference in the proportion of Cs and Ds in the neighborhood, there will be a tendency to imitate the most frequent strategy. In subsequent updates, the agent will tend to keep its strategy since the most part of its neighbors has the same strategy it has. This contributes to less strategy changes and, hence, less instability in the system. On the other side, situations where there is an equilibrium between Cs and Ds in the neighborhood will lead, on average, to more uncertainty in the strategy to imitate. This leads to more strategy changes, which means more instability.

In what concerns the dependency of the instability level on the interaction topology, we found that, for SFNs, instability grows as m is increased (see example in Fig. 5). This is a natural result since the larger m the more links exist between the agents and, therefore, the greater the potential for strategy changes. For SWNs, we confirmed the results of Abramson and Kuperman [1] for the PD game: instability grows with ϕ. This effect is, however, absent in the SD game.

Finally, Figs. 2 and 3 also illustrate that instability is bigger for the SD game than for the PD game. As stated in Section 2.1, in the SD game, it is better to play C if the other player plays D and vice-versa. On the other side, playing D is always the best option in the PD game. This means that a C agent in the SD game is more resistant to the presence of D neighbors. In general, this leads to the formation of smaller and less compact clusters in the SD game than in the PD game. This, in turn, means that, on average, there are more C-D links in a population of SD players than in a population of PD players, which is a possible explanation for more instability in the SD game.

4 Conclusions and Future Work

In this work we identified the features that determine the instability of the Spatial Prisoner's Dilemma and Snowdrift games. Stability is here defined as the proportion of strategy changes that happen in the asymptotic period of the evolutionary process. As far as we know, this is one of the first works where such issue is investigated in the context of spatial evolutionary games. Besides the two games, a big number of conditions were tested: different interaction topologies, noise levels and synchrony rates were used.

The results can be summarized in the following way: As the noise level is increased, the instability becomes increasingly dependent on the composition of the population only. More specifically, instability increases as the proportion of cooperators in the population approaches 0.5. This means that in situations where there is a significant noise level, the identification of defecting agents becomes more difficult as the proportions of cooperating and defecting agents are similar. On the other side, for very small noise values, the instability decreases as the synchrony rate is decreased in the Prisoner's Dilemma game. In the Snowdrift this happens only for random networks and in scale-free networks for synchrony rates bellow $\alpha = 0.8$.

Trying to explain the effect of the synchrony rate for small noise values in both games and the behavior differences between them in this regime will be one of the main future directions of this work. In further extensions we will also explore other transition rules. The generalized proportional transition rule used in this work models a situation of complete neighborhood monitoring. This means that each agent analyses the payoffs of all its neighbors in order to decide which strategy to adopt next. As the noise level increases, the relative proportion of cooperators and defectors in the neighborhood assumes a relevant role in the decision process. This aspect is determinant in the instability level for significant noise values. Hence, it makes sense to experiment with transition rules that model just a partial neighborhood monitoring, while allowing variable noise levels. The *sigmoid* transition rule [15] is a good candidate for this test because it allows varying degrees of noise and only the payoff of one neighbor, randomly selected, is taken into account in the decision process. Finally, we will also explore n-player games such as the Public Goods game in order to verify if the results achieved so far are generalizable to these situations.

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